3.2 Addition and Subtraction

If set R has r elements and set S has s elements, and R and S are disjoint, then the sum of r and s, written $r + s$, is the number of elements in the union of R and S. The numbers r and s are called **addends**.

-The result of an addition is called a **sum** or a **total**. The parts of the sum are **addends** or **terms.** Addition (and the other three elementary operations) are called **binary operations** because they involve two numbers. Even when an entire column of numbers is added, numbers are always added two at a time.

Disjoint Sets

-The two sets involved in addition must be disjoint. If sets intersect, the number of elements in their union is not the sum of the elements in the two sets. For instance, consider a class in which 16 are students are enrolled in Calculus and nine are also band members. The number of students taking Calculus or band could be less than 25 because some Calculus students may also be a band member.

Common Units

-While theoretically we can take the union of any two disjoint sets, in practice we only do so if the elements of these sets share a significant characteristic. For instance, it is unlikely that anyone would add three bananas to five chairs, though theoretically the sum would be eight objects. Sums that contain more than one unit are **mixed numbers**. To express such a sum in terms of one number, a conversion needs to be made to a common unit.

Situations Involving Addition:

1) Patrick and Ian bought a pizza cut into eight equal pieces. Patrick ate three eighths, and Ian ate four eighths. How much did they eat altogether?
2) Bobby was 4'7" high last year, and this year he grew another 2". How tall is he now?
3) On the first day of her backpacking trip, Rosemary ate four Tootsie Rolls from her pack. That night, she did an inventory and found that she had 14 Tootsie Rolls left. How many Tootsie Rolls did she have originally?
4) Billy is 4" taller than Eddie. Eddie is 3"8". How tall is Billy?
5) Fourteen students in Ms. Long's homeroom are enrolled in Spanish, while eight are signed up for piano lessons. How many of her students are studying Spanish or taking piano lessons?

Properties of Addition

1) Identity Property of Addition: $a + 0 = 0 + a = a$
2) Commutative Property of Addition: $a + b = b + a$
3) Associative Property of Addition: $(a + b) + c = a + (b + c)$
4) Closure Property of Addition: The sum of any two whole numbers is a unique whole number. There are two parts to this property: (1) uniqueness (the sum will always be the same number) and (2) the sum will always be a whole number.

Not all sets are closed under addition. For example, consider the set of odd numbers.

Models for Addition Algorithms

An **algorithm** is a step-by-step procedure for computing. Algorithms for addition involve two separate procedures: (1) adding digits and (2) regrouping, or "carrying" (when necessary), so that the sum is written in positional numeration. These two methods use an algorithm called **partial sums.**

Example 1: (A) Adding digits (B) Regrouping

	6	4	7
+	1	7	9
		1	6
	1	1	
	7		
	8	2	6

	6	4	7	=	6	hundreds	+	4	tens	+	7
+	1	7	9	=	1	hundreds	+	7	tens	+	9
					7	hundreds	+	11	tens	+	16
Re	gro	upi	ng:		8	hundreds	+	2	tens	+	6
				=	826						

Left-to-Right Addition

Example 2: Compute the sum 897 + 537 using left-to-right addition.

Solution: To compute 897 + 537 from left to right, we first add 8 and 5 in the hundreds column. In the second step, 9 and 3 are added in the tens column, and because regrouping (carrying) is necessary, 3 in the hundreds column is scratched out and replaced by 4. In the third step, we add the units digits. Again regrouping is necessary, so 2 in the tens column is scratched out and replaced by 3.

The early Hindus and later the Europeans added from left to right. The Europeans called this algorithm the **scratch method**.

An Alternative Algorithm

When base ten was invented, several different algorithms for each operation were invented as well. One alternative algorithm for addition is the **lattice algorithm.** First, write the problem. Below each place draw a square. Second, draw diagonal lines through each square that extend below the square. Third, write the result of each partial sum in the box. Last, add diagonally. If the sum in any diagonal addition is greater than 10, ”carry” just as you do with the standard algorithm.

Example 3: Use the lattice algorithm to add: A) 645 + 728 B) 5462 + 1209

Number Sense and Operation Sense

We want to learn more than simply how to use the various algorithms efficiently and accurately. We want to move from merely understanding how to also understanding why. A big part of this transformation involves developing what is called number sense and operation sense. Number and operation sense involves the ability:

1) to take numbers apart and put them together
2) to move fluently among different representations
3) to recognize when one representation is more useful than another
4) to perform mental computation and estimation flexibly
5) to understand the effect of different operations on numbers

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Subtraction

The abstract model for subtraction is the same as for addition, two disjoint sets and their union. However, in subtraction, one of the parts is unknown, not the whole.

Basic Definition of Subtraction: Subtraction is the process of finding the number of elements in one of two disjoint sets, when the union and the other set are of known size.

Subtraction reverses the action of addition. That is, the act of adding a part can be ”undone” by subtracting the part. Formally, we say subtraction is the **inverse** of addition.

The result of a subtraction is called a difference. Because the order of the two numbers in a subtraction is significant, they are given different names. If $a - b = c,$ then a is called the **minuend**, b is called the **subtrahend**, and c is called the **difference**.

Situations Involving Subtraction

1. Part-Part-Whole (Unknown: a part)

-involves either a physical or mental combination of sets of more or less equal standing

Example: A class of 24 students includes 10 boys. How many girls are in the class?

Solution: $10 + x = 24$ or $24 - 10 = x; x = 14$

2. Adding On

-involves an original quantity, a quantity that is added, and a new quantity

a) Unknown: adding part - - missing addend

Example: Becca is saving her allowance to buy a CD that costs $18. She has saved $10 so far. How much more does Becca need to save?

Solution: $10 + x = 18$ or $18 - 10 = x; x = 8$

b) Unknown: original quantity

Example: A teacher sent some of her class to the library and later sent six students to a special math tutoring session. Eighteen students are now out of the classroom. How many students were sent to the library?

Solution: $x + 6 = 18$ or $18 - 6 = x; x = 12$

3. Separation

-starts with a whole quantity from which a part is taken away

a) Unknown: remaining part

Example: A teacher sent 25% of her students to the library. What percent of her class stayed in her classroom?

Solution: 100% of the class – 25% of the class $= x; x =$ 75% of the class

b) Unknown: the part taken away

Example: Twenty four students were in a class at the beginning of the semester. At the end of the semester, only 20 students remained. How many dropped the course?

Solution: $24 - 20 = x; x = 4$

4. Comparison

a) Unknown: difference

Example: Keanna is 5’8” and Darryl is 6’2”. How much taller is Darryl than Keanna?

Solution: 6’2”– 5’8” $= x; x =$ 6”

b) Unknown: smaller quantity

Example: Darryl, who is 6’2”, is 6 inches taller than Keanna. How tall is Keanna?

Solution: 6’2”–$x =$ 6”; $x = 5'8$”

Special Case of Comparison: Distance on a Number Line

Example: One exit on an interstate is at mile marker 34 and another is at mile marker 51. How far apart are these exits?

(This is an abstract application of subtraction because the example involves points, not sets or lengths. To ”see” the latter requires relating points to their directed distances from 0.)

All of the above subtraction situations have the same fundamental structure: two disjoint sets, one known and one unknown, and their known union. Students must be taught to look for this structure, especially since it is sometimes obscure. Students also need to understand the relation between addition and subtraction so they can transform sentences with missing addends into corresponding subtraction sentences and vice versa.

Properties of Subtraction

1) Identity Property of Subtraction: The operation of subtraction does not have an identity property. Although $a - 0 = a,$ the reverse is not true–that is, $0 - a \neq a.$ We say that the operation of subtraction has a right-identity but not a left-identity.

2) Commutative Property of Subtraction: The operation of subtraction does not possess the commutative property.

3) Associative Property of Subtraction: The operation of subtraction does not possess the associative property.

4) Closure Property of Subtraction: The operation of subtraction is not closed for the set of whole numbers because the difference of two whole numbers can be a negative number.

Example 1: Determine the minimum number of flats, longs, and units for the bases if the pieces in set A are combined with the pieces in set B. Then write numbers in positional numeration for sets A and B and the number for their sum in the given base.

A) Base five
- A: 3 flats, 2 longs, 4 units
- B: 2 flats, 1 long, 3 units

B) Base twelve
- A: 9 flats, 5 longs, T units
- B: E flats, 4 longs, 2 units

Example 2: Determine the minimum number of flats, longs, and units for the bases needed to be combined with set B to obtain set A for the given base. Then write numbers in positional numeration for sets A and B and the number for their difference in the given base.

A) Base eight
- A: 6 flats, 3 longs, 2 units
- B: 4 flats, 7 longs, 6 units

B) Base ten
- A: 9 flats, 8 longs, 4 units
- B: 7 flats, 4 longs, 5 units