Math 205 **Section 4.1 Factors and Multiples**

Problem Opener: Fifty pennies are placed side by side. Each second penny is replaced by a nickel, each third coin is replaced by a dime, each fourth coin is replaced by a quarter, and each fifth coin is replaced by a half-dollar. What is the value of the 50 coins?

Number theory is the study of nonzero whole numbers and their relationships. One important type of relationship in number theory is that between a factor and a multiple. If one number is a factor of a second number or divides the second, then the second number is a multiple of the first.

FACTOR AND MULTIPLE If a and b are whole numbers and $a \neq 0,$ then a is a factor of b if and only if there is a whole number c such that $ac = b.$ We can say that a divides b or that b is a multiple of $a.$ If a and b are whole numbers such that a divides b (a is a factor of b), we write $a \mid b.$ If a does not divide $b,$ we write $a \nmid b.$

Example 1: To become more familiar with the divides relationship, classify the following statements as true or false.

A) $15 \mid 60$ B) $8 \mid 30$ C) $3 \nmid 19$ D) $18 \mid 18$ E) $2 \mid 0$

Notice that divides signifies a relationship between two numbers; it indicates that one number is divisible by another. It does not indicate the operation of division, that is dividing one number by another. For example, $3 \mid 15$ tells us that 3 divides 15 and should not be confused with the fraction $\frac{3}{15}$, which means 3 divided by 15 and is equal to the fraction $\frac{1}{5}$.

DIVISIBILITY PROPERTIES

1. If a number divides each of two other numbers, then it divides their sum. If $a \mid b$ and $a \mid c,$ then $a \mid (b+c).$
2. If a number divides one of two numbers but not the other, then it will not divide their sum. If $a \mid b$ and $a \nmid c,$ then $a \nmid (b+c).$
3. If one number divides another number, then it will divide the product of that number with any other whole number, $k.$ If $a \mid b,$ then $a \mid kb.$

Divisibility Test	Example
1) A number is divisible by 2 if the last digit is 0, 2, 4, 6 or 8.	1) 168 is divisible by 2 since the last digit is 8.
2) A number is divisible by 3 if the sum of the	2) 168 is divisible by 3 since the sum of the digits
digits is divisible by 3.	is 15 (1 + 6 + 8 = 15), and 15 is divisible by 3.
3) A number is divisible by 4 if the number formed	3) 316 is divisible by 4 since 16 is divisible by 4.
by the last two digits is divisible by 4.	
4) A number is divisible by 5 if the last digit is either 0 or 5.	4) 195 is divisible by 5 since the last digit is 5.
5) A number is divisible by 6 if it is divisible by 2	5) 168 is divisible by 6 since it is divisible by 2
AND it is divisible by 3.	AND it is divisible by 3.
6) A number is divisible by 8 if the number formed	6) 7,120 is divisible by 8 since 120 is divisible by 8.
by the last three digits is divisible by 8.	
7) A number is divisible by 9 if the sum of the	7) 549 is divisible by 9 since the sum of the digits is
digits is divisible by 9.	18 (5 + 4 + 9 = 18), and 18 is divisible by 9.
8) A number is divisible by 10 if the last digit is 0.	8) 1,470 is divisible by 10 since the last digit is 0.

Divisibility by 7

To determine if a number is divisible by 7, take the ones digit off of the number, double the ones digit and subtract the doubled number from the remaining number. If the result is evenly divisible by 7 (e.g. 14, 7, 0, -7, etc.), then the number is divisible by seven. This process may need to be repeated several times.

Example 2: Determine if 3101 is divisible by 7.

Example 3: Classify each statement as true or false and explain why.

A) $4 \mid 113,775,920$

B) $5 \mid 3,135$

C) $2 \mid 2,461$

D) $9 \mid 147,389$

E) $6 \mid 2,100,000,472$

Prime and Composite Numbers

An integer greater than one is called a prime number if its only positive divisors (factors) are one and itself. Nonzero whole numbers with more than two factors are called composite numbers. Since 1 has only one factor, it is classified as neither prime nor composite.

There is no largest prime because there are an infinite number of prime numbers. Some very large primes have been discovered. From 1876 to 1951, this 39-digit number was the largest known prime:

170,141,183,460,469,231,731,687,303,715,884,105,727.

Now computers make it possible to find a larger prime every few months. In November 2001, for example, the team of Michael Cameron, George Woltman, Scott Kurowski, et al. discovered a prime number with over 4 million digits. Prime numbers are difficult to locate because they do not occur in predictable patterns. In fact, there are arbitrarily large stretches of consecutive whole numbers that include no primes. For example, between the numbers 396,733 and 396,833 there are 99 composite numbers.

Historical Highlight: Knowing the factors of numbers have long been valuable in research involving prime numbers. In 1659, J. H. Rahn published a table listing the factors for all numbers up to 24,000, and in 1668, John Pell of England extended this table to 100,000. The greatest achievement of this sort is the table by J. P. Kulik (1773-1863) from the University of Prague. His table covers all numbers up to 100,000,000. Finding the factors of numbers is currently of major interest to cryptographers and intelligence agencies, whose code solutions are often based on the prime factors of very large whole numbers.

(H. W. Eves, An Introduction to the History of Mathematics, 3rd ed. (New York: Holt, Rinehart, and Winston, 1969), p. 149)

PRIME NUMBER TEST

Suppose n is a whole number and k is the smallest whole number such that $k \times k$ is greater than n.

If there is no prime number less than k that is a factor of n, then n is a prime number.

To determine if a number has factors other than itself and 1, we need only try dividing by prime numbers. There is no need to divide by composite numbers. For example, if 4 divides a number, then 2 divides the number. In other words, if 2 does not divide a number, then 4 will not divide the number.

Example 4: Which of the following numbers are prime?

A) 43,101

B) 24,638

C) 53

Example 5: A) Is 421 prime or composite?

B) Is 667 prime or composite?