Drawing Logical Conclusions

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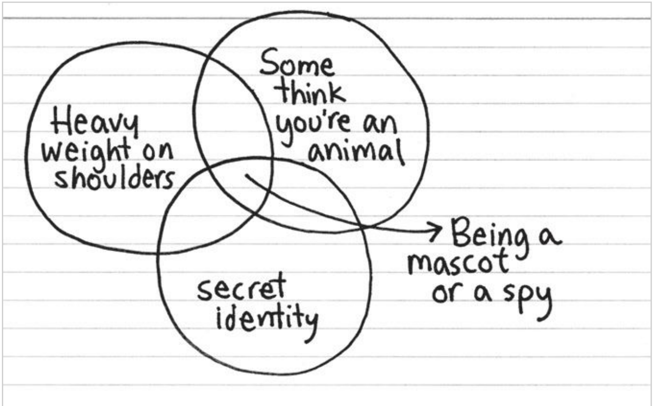
Venn diagrams are everywhere. A web search of "Venn diagram" produces articles as diverse as those teaching ethical reasoning, relating psychiatric symptoms, introducing instructional strategies for pedagogy, and training in agricultural education—all of them using the familiar overlapping circles. They are even a favorite source for Jessica Hagy's insightful humor on her blog, *Indexed* (see figure 1).



Jessica Hagy, http://thisisindexed.com/2013/11/representing-your-team/

Figure 1. "Representing your team."

Venn diagrams are a standard part of the curriculum of introductory set theory. Figure 2 shows the well-known depictions of the intersection, union, and complement of sets. But surprisingly, Venn diagrams were not originally created to study sets, nor was John Venn the first to draw such diagrams. They were introduced to help analyze logical arguments; the application to set theory became popular in the mid-20th century. And similar diagrams were used at least two centuries before Venn wrote about them.

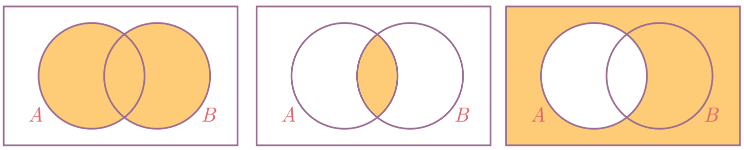
Rules of logical arguments originated with Aristotle circa 300 BC. He was mainly concerned with statements or propositions about categories (not sets). If we have two categories, A and B, the four types of propositions are: all A are B; no A is B; some A are B; and some A are not B. Aristotle demonstrated how to put propositions together to create a valid argument, called a syllogism. There is no evidence that Aristotle used circle diagrams to represent these relationships.

After Aristotle's era, there were a few instances of circle diagrams showing relationships between categories, but the use of diagrams to illustrate the validity of Aristotelian syllogisms was first brought to the attention of the public by the Swiss mathematician Leonhard Euler.

Euler used the diagrams as a teaching tool in letters later published under the title *Letters to a German Princess.* Written in 1761 and published in 1768, *Letters* contained Euler's portion of a two-year correspondence with the 15-year-old cousin of Frederick the Great. Euler had been asked to tutor the princess and her younger sister in such diverse fields as physics, philosophy, mechanics, astronomy, optics, and acoustics. Euler's circle diagrams were part of his instructions to the princess in Aristotle's logic of the syllogism.

Most historians of mathematics are aware of Euler's contributions, but few know that virtually identical circles appeared nearly a century earlier in the 1686 logic papers of the universal genius Gottfried Leibniz. However, Leibniz's circles were published in 1903, long after his (and Euler's) death. They can be found in the short paper "*De Formae Logicae Comprobatione per Linearum Ductus*," in a collection of unedited fragments of his work on logic.

It is unknown whether Euler had seen Leibniz's diagrams. If he did know of them, it may have been through the colleagues and friends the two men had in common (the intersection of Leibniz's colleagues and Euler's colleagues). During the 17th and 18th centuries, letters were a semi-public form of communication, with the sender sometimes requesting that the receiver share



$A \cup B$ Union | $A \cap B$ Intersection | $\sim A$ Complement

Figure 2. Typical modern examples of the use of Venn diagrams in set operations.

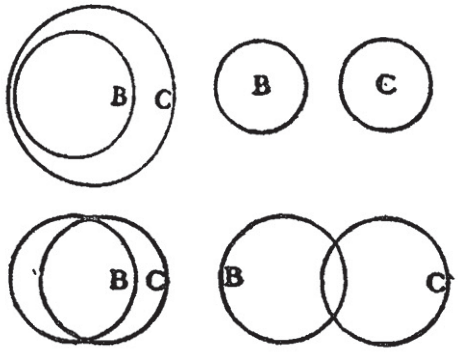


Figure 3. Leibniz's circle diagrams, not published until 1903. Clockwise from top left: all *B* are *C*, no *B* is *C*, some *B* are *C*, and some *B* are not *C*.

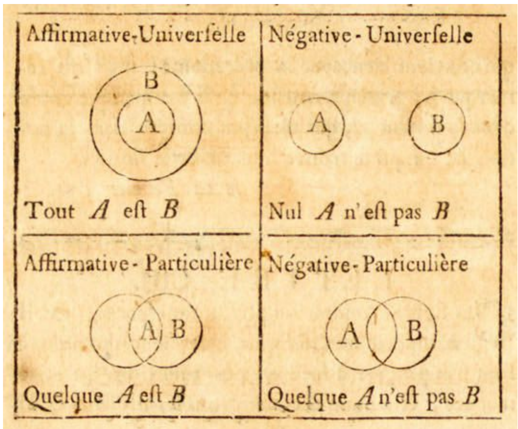


Figure 4. Euler's circle diagrams from a 1770 publication of *Letters*. Clockwise from top left: all *A* are *B*, no *A* is *B*, some *A* are *B*, and some *A* are not *B*.

the letter with other individuals.

Libraries that house Leibniz's manuscripts estimate that the documents include about 15,000 letters that he wrote to about 1,100 correspondents. Among his correspondents were Johann Bernoulli, who was a mentor and dissertation adviser to Euler, and Nicolaus Bernoulli (Johann's nephew), who corresponded with both Leibniz and Euler. But the direct link between Leibniz's circles and Euler's circles—if there is one—has yet to be uncovered. Of course, the circle diagrams may not be original with Leibniz either.

Figures 3 and 4 illustrate Leibniz's and Euler's diagrammatic representations of the four types of Aristotelian propositions. Other than their choices of labels for the categories, the diagrams are almost identical. The "all" proposition is understandable; if all of one thing is included in another, then the category that is encompassed should be a circle inside the other circle. Today we might think of that relationship as A is a subset of B (in the case of Euler) or B is a subset of C (in the case of Leibniz). The "no" propositions are also obvious (we would say that the sets are disjoint).

It is interesting that both men used the category labels in the same way in the "some are" and "some are not" propositions. They placed the labels (A, B, or C) inside the intersection of the circles to indicate when "some are" and outside the intersection of the circles to indicate when "some are not."

This turns out to be extremely awkward notation for the "some are not" propositions. There is no difference between the diagrams for "some A are not B" and "some B are not A" even though they are not equivalent statements. "Some dogs are not poodles" is true, while "some poodles are not dogs" is not. Although both Euler and Leibniz suggested this convention, they were inconsistent in maintaining it. (Euler later changed notation, using an asterisk [*] instead of a category label to designate the compartment where "some" should be included.)

Let's see how Euler used his diagrams to analyze a syllogism; Leibniz's approach is similar. We give an example from Euler's *Letters*, which is doubly interesting because he made an error. He presented the princess with the following valid syllogism, in which the first two propositions are the premises (accepted as true) and the third-line conclusion necessarily follows from those premises.

No virtuous man is a slanderer.
But some slanderers are learned men.
Therefore, some learned men are not virtuous.

Using the symbols A, B, and C for the categories of virtuous men, slanderers, and learned men, respectively, this becomes:

No A is B.
But some B are C.
Therefore, some C are not A.

Figure 5 shows Euler's diagrams. He realized that there are three possible drawings that can be inferred

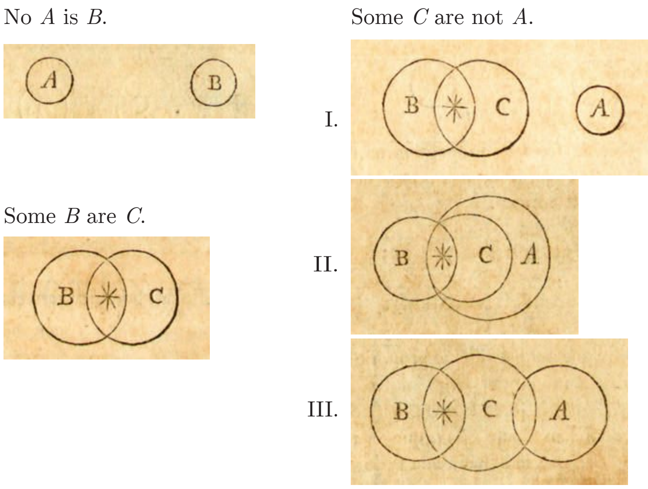


Figure 5. Euler's analysis of the syllogism.

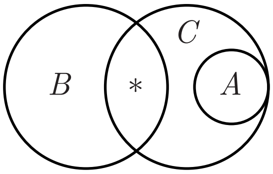


Figure 6. Euler's case II corrected.

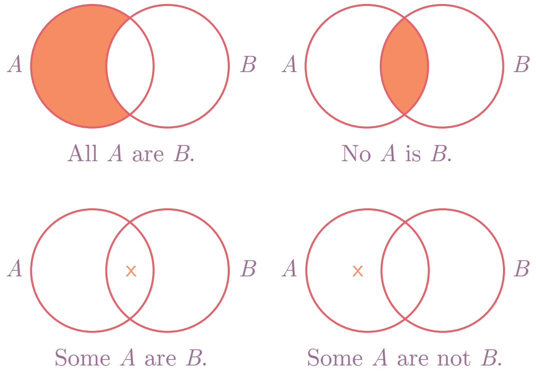


Figure 7. Aristotle's four categorical propositions in Venn's diagram.

from the two premises—all of which support the conclusion. The three cases should be that space A is excluded from the non-star portion of space C, wholly contained in non-star C, or partially contained in non-star C. Euler should have drawn case II as in figure 6.

Even if Euler's solution was correct, the three-case conclusion is tedious. As we shall see, Venn's circle diagrams make this problem of needing three cases disappear.

Between the time that Euler's *Letters* appeared and the development and study of symbolic logic in England by George Boole and Augustus De Morgan, the circle diagrams were used extensively and became known as Euler's circles. When John Venn published his ideas about diagrams in logic in 1880, he mentioned that of 60 logical treatises published during the last century he had (rather haphazardly) consulted, 34 had appealed to the use of diagrams, nearly all making use of the Eulerian scheme. Venn did not mention Leibniz's use of similar diagrams.

The main difference between the Euler/Leibniz circles and Venn's new diagrams is that in the Euler/Leibniz method we are required to know the relationships between categories or classes before we can draw the diagrams—sometimes the circles overlap; sometimes one is entirely inside another; and sometimes they do not touch. Venn's method begins each time with the same diagram.

For example, for two terms, Venn always used two overlapping circles. He then added shading or markings to describe the proposition, as in figure 7. A shaded compartment indicated that there was nothing in that

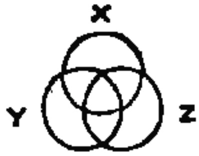


Figure 8. Venn's 1880 template for three terms.

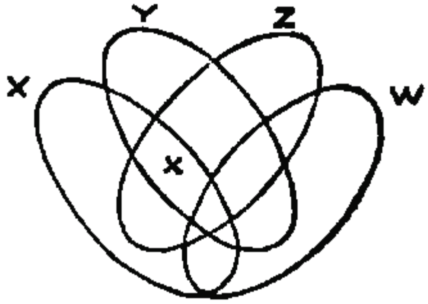


Figure 9. Four-term logic diagram by John Venn in 1880.

region (the set is empty). This, by the way, is different from the way we use shading today—to draw attention to a region of interest. An x in a compartment meant that there was at least one item in that region. Nothing could be said about the status of an unshaded region without an x.

Venn's two overlapping circles created three disjoint compartments, and the region outside the two circles created a fourth compartment. These correspond to the four combinations: both A and B, A alone (A and not-B), B alone (not-A and B), neither A nor B (not-A and not-B). Today, we generally put a large rectangle around the circles to rein in the outside region, but Venn saw no necessity for that.

Eight regions are needed for the three terms involved in the two-premise syllogisms. Venn's three-term diagrams would indicate inclusion (the x) or exclusion (the shading) on a picture of three overlapping circles as seen in his template in figure 8.

Another feature of Venn's system that made it superior to the previous systems was that diagrams could easily be invented that allowed more than three terms. Constructing the diagrams is an interesting mathematics problem in itself. If there are n categories, then the diagram would require n circles—or other shapes—overlapping to form 2^n regions. For four terms Venn used ellipses as in figure 9. There are 16 compartments, including the outside region.

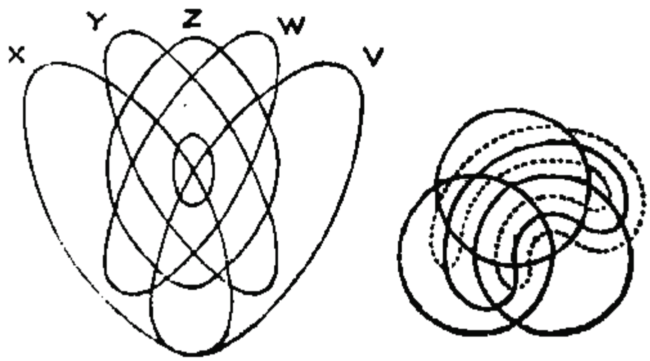
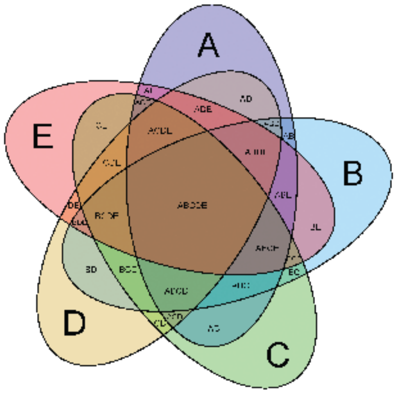
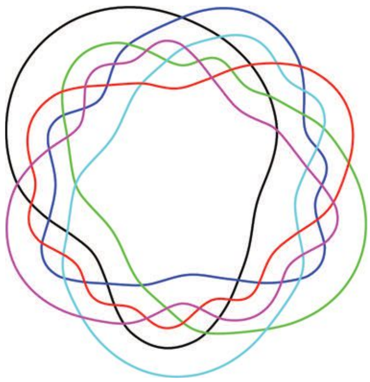


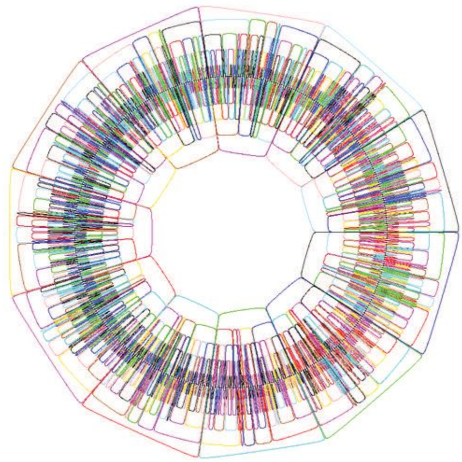
Figure 10. Two five-term diagrams by John Venn.



(a) http://commons.wikimedia.org/wiki/File:Symmetrical_5-set_Venn_diagram.png, Branko Grünbaum



(b) http://webhome.cs.uvic.ca/~ruskey/Publications/SixVenn/SixVenn.html "Generating all simple convexly-drawable polar symmetric 6-Venn diagrams," 2011, by Khalegh Mamakani and Frank Ruskey



(c) Peter Hamburger *newscientist.com/gallery/venn/8*

Figure 11. (a) 5-set, (b) 6-set, and (c) 11-set Venn diagrams.

Venn could not find an alignment of ellipses that would work for five terms, unless one of them had a hole in it (a shape called an annulus) or unless two of the regions were horseshoe-shaped. Venn's five-term, 32-compartment diagrams are shown in figure 10.

Today mathematicians are interested in developing Venn diagrams for larger numbers of sets, often looking for those with beautiful symmetries. Figure 11 illustrates 5-, 6-, and 11-set Venn diagrams containing 32, 64, and 2,048 regions, respectively!

In figure 12 we see how Venn would have used his diagrams to analyze the validity of Euler's syllogism. The first premise is that no A is B, so we shade the intersection of A and B. The second premise states that some B are C, so we put an x in the nonshaded region shared by B and C. Because the x is in C, but not in A, we conclude that some C are not A. Notice that unlike Euler's analysis, Venn's does not require cases.

Martin Gardner, author of the "Mathematical Games" column in *Scientific American* for more than 20 years, called Venn's diagrams an "ingenious improvement on Euler's circles. The result was a diagrammatic method so perfectly isomorphic with the Boolean class algebra, and picturing the structure of class logic with such visual clarity, that even a nonmathematically minded philosopher could 'see' what the new logic was about."

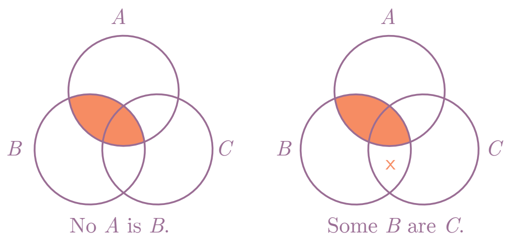


Figure 12. Using Venn diagrams to illustrate Euler's syllogism.

The complete history of circle diagrams has yet to be uncovered. Euler influenced Venn. Leibniz may have influenced Euler. And Leibniz's predecessors may have influenced him. Until we find out the full story, we celebrate the reality that the origins of mathematical concepts continue to intrigue, challenge, and mystify.

Further Reading

Deborah J. Bennett, *Logic Made Easy: How to Know When Language Deceives You* (New York: W. W. Norton & Company, 2004).

Leonhard Euler, *Letters of Euler on Different Subjects in Physics and Philosophy Addressed to a German Princess* (London: Murray and Highley, 1802).

Martin Gardner, *Logic Machines and Diagrams* (New York: Mcgraw-Hill, 1958).

Gottfried Wilhelm Leibniz, *"De Formae Logicae Comprobatione per Linearum Ductus," Opuscules et fragments inédits de Leibniz*, ed. Louis Couturat (Paris: Félix Alcan, 1903): 292–331.

John Venn, "On the diagrammatic and mechanical representations of propositions and reasonings," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 9, no. 59 (1880): 1–18.

John Venn, *Symbolic Logic* (London: Macmillan and Co., 1881).

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